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SPACE CONSERVATION LAW IN FINITE VOLUME CALCULATIONS OF FLUID FLOW

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SUMMARY

In the numerical solutions of fluid flow problems in moving co-ordinates, an additional conservation equation, namely the space conservation law, has to be solved simultaneously with the mass, momentum and energy conservation equations. In this paper a method of incorporating the space conservation law into a finite volume procedure is proposed and applied to a number of test cases. The results show that the method is efficient and produces accurate results for all grid velocities and time steps for which temporal accuracy suffices. It is also demonstrated, by analysis and test calculations, that not satisfying the space conservation law in a numerical solution procedure introduces errors in the form of artificial mass sources. These errors can be made negligible only by choosing a sufficiently small time step, which sometimes may be smaller than required by the temporal discretization accuracy.

KEY WORDS Space Conservation Numerical Calculation Fluid Flow

1. INTRODUCTION

In fluid flow calculations the use of moving co-ordinates is sometimes essential, e.g. in flows with moving boundaries. The solution of such problems is best accomplished via the conservation equations in a non-Eulerian co-ordinate frame. Owing to the movement of the co-ordinate system, an additional equation results which has to be satisfied simultaneously with the other conservation equations. This equation relates the change of the elementary control volume to the co-ordinate frame velocity and is hence called by Trulio and Trigger[1] the 'space conservation law' (SCL). They seem to be the first to have included this equation together with the mass, momentum and energy transport equations in their 'fundamental equations of motion' for numerical solutions on moving meshes; they used it for one-dimensional flow calculations. However, the necessity of solving this equation simultaneously with the other conservation equations was not recognized until it was rediscovered by Thomas and Lombard[2] and by Demirdžić;[3] Warsi[4] also recognizes the SCL equation as 'the fundamental equation for non-steady co-ordinates'. The only applications of the SCL known to the authors are those of Trulio and Trigger[1] and of Thomas and Lombard,[2] who used it in conjunction with a finite difference solution method.

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A number of flow calculations on moving meshes are reported in the literature. However, none of them, apart from those mentioned above, uses the SCL. In most cases either the authors were fortunate that their choice of the grid velocity satisfied the SCL (e.g. Gosman and Watkins,[5] Gosman and Johns[6]) or the error coming from the non-satisfaction of the SCL was attributed to other sources. For instance, Viviand and Ghazzi[7] encountered oscillations and instabilities in their finite difference calculations. In order to overcome these problems, they abandoned the conservative in favour of a non-conservative form of the equations which does not contain the Jacobian. Similarly, Amsden *et al.*[8] reported that their finite volume calculations were excessively sensitive to the volume changes for subsonic flows and difficulties were encountered in the pressure iterations.

In the next section we discuss the importance of the space conservation law in numerical calculations with moving grids. By employing finite volume (FV) method and fully implicit temporal differencing, we propose in Section 3 a way of calculating the grid velocities so that the SCL is automatically satisfied. On test calculations reported in Section 4 we demonstrate the effects of errors introduced by not satisfying the SCL. Finally the conclusions from this exercise are drawn up in Section 5.

2. SPACE CONSERVATION LAW

The set of equations describing conservation of space, mass, momentum and energy in a moving co-ordinate frame reads respectively[3,4]

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}-\operatorname{div}\mathbf{v}_{g}=0, \tag{1a}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}(\sqrt{g}\rho)+\operatorname{div}(\rho\mathbf{v}_{r})=0, \tag{1b}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}(\sqrt{g}\rho\mathbf{v})+\operatorname{div}(\rho\mathbf{v}_{r}\mathbf{v}-\mathbf{T})=\mathbf{S}_{v}, \tag{1c}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}(\sqrt{g}\rho\phi)+\operatorname{div}(\rho\mathbf{v}_{r}\phi-\mathbf{q})=S_{\phi}. \tag{1d}$$

Here $\sqrt{g}$ is the determinant of the metric tensor, $\mathbf{v}$ is the velocity vector, $\mathbf{v}_r=\mathbf{v}-\mathbf{v}_g$ is the fluid velocity relative to the moving co-ordinate system (grid), $\mathbf{v}_g$ is the grid velocity and ϕ is a scalar quantity (temperature, concentration, etc.). $\mathbf{T}$ and $\mathbf{q}$ are the stress tensor and the scalar flux vector respectively given by

$$\mathbf{T}=-(p+\tfrac{2}{3}\mu\operatorname{div}\mathbf{v})\mathbf{I}+2\mu\mathbf{D}, \tag{2a}$$

$$\mathbf{q}=\Gamma_{\phi}\operatorname{grad}\phi, \tag{2b}$$

where p is the pressure, μ is the dynamic viscosity, $\mathbf{I}$ is the unit tensor, $\mathbf{D}$ is the deformation rate tensor (the symmetric part of the velocity gradient) and Γ_{ϕ} is the diffusivity of ϕ. The terms on the right-hand sides of equations (1) represent sources or sinks.

In order to demonstrate the necessity of satisfying the SCL, let us consider incompressible fluid (ρ=constant). In this case the mass conservation equation (1b) can be rewritten as

$$\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}-\operatorname{div}\mathbf{v}_{g}\right)+\operatorname{div}\mathbf{v}=0. \tag{3}$$

The bracketed terms in equation (3) represent the left-hand side of the SCL equation (1a) and must be zero. Thus the mass conservation equation for the incompressible fluid reads

$$\operatorname{div} \mathbf{v} = 0. \tag{4}$$

If the SCL is not satisfied by the numerical solution procedure, artificial mass sources are generated which may cause the solution to be greatly in error. This will be demonstrated later by the test calculations performed for a stationary fluid and moving grids.

Equation (1a) or equivalently equation (3) could be used to calculate the change in cell volume, δV, for a given grid velocity $\mathbf{v}_g$. However, this approach is not practical since the grid velocity is generally not available. Moreover, in most practical applications, for example in-cylinder flow and flow around moving valves, the grid position at each time level is prescribed and the change in the cell volume as well as the surface vectors are given (known) quantities.

Unlike Thomas and Lombard,[2] who *solve* equation (1a) numerically together with the other conservation equations, we propose to *calculate* the grid velocities from the known grid positions in a manner consistent with the discretization of the other conservation equations, so that the SCL is always exactly satisfied. This will be outlined in the next section. Attention will be paid only to the SCL; discretization of other conservation equations is described in detail in Perić.[9] A fully implicit scheme is employed, for reasons discussed by Demirdžić[3], but the practice proposed will apply to other kinds of temporal differencing as well.

3. DEFINITION OF THE GRID VELOCITIES

When equation (1a) is integrated over an arbitrary control volume (cell) and time, with the aid of the Gauss' divergence theorem, one gets

$$\frac{1}{\Delta t}(V^{\mathrm{n}} - V^{\mathrm{o}}) = \oint_S \mathbf{v}_g \cdot \mathbf{dS}^{\mathrm{n}}, \tag{5}$$

where $\delta V = V^{\mathrm{n}} - V^{\mathrm{o}}$ is the change of the cell volume during Δt, S is the surface of the control volume, $\mathbf{dS}$ is the surface vector and superscripts 'n' and 'o' denote the new and old time levels respectively.

For Cartesian velocity components and an arbitrary quadrilateral control volume (see Figure 1), equation (5) becomes

$$\frac{1}{\Delta t}(V^{\mathrm{n}} - V^{\mathrm{o}}) = \sum_i \mathbf{v}_{gi} \cdot \mathbf{S}_i^{\mathrm{n}}, \quad i = \mathrm{e, w, n, s}, \tag{6}$$

where $\mathbf{v}_{gi} = (u_g, v_g)_i$ is the cell face velocity and $\mathbf{S}_i = (S_x, S_y)_i$ is the cell face vector.

In order to satisfy the SCL, we define the grid velocities so that the rate of change of the cell volume obtained from the SCL, $(\delta V_{\mathrm{SCL}})/\Delta t$, is exactly equal to its actual (geometrical) rate of change, $(\delta V_{\mathrm{G}})/\Delta t$, i.e.

$$\delta V_{\mathrm{G}} \equiv \delta V_{\mathrm{SCL}}. \tag{7}$$

This will be applied first to a simple one-dimensional case and then to control volumes of gradually increasing geometrical complexity.

One-dimensional grid

In one-dimensional case the natural definition of the cell face (average) velocities is (see Figure 2)

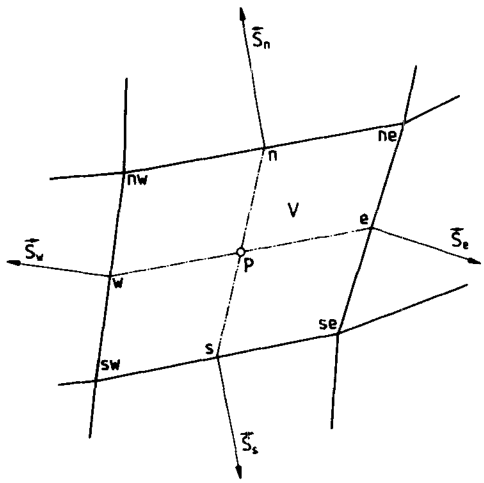

Figure 1. Control volume and labelling scheme

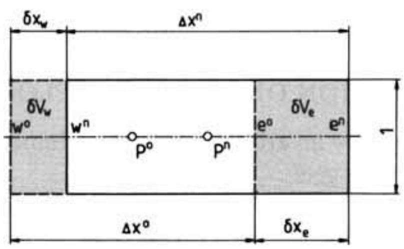

Figure 2. Control volume in one-dimensional moving grid at two time levels

$$u_{g,e}=\frac{\delta x_e}{\Delta t}, \qquad u_{g,w}=\frac{\delta x_w}{\Delta t}. \tag{8}$$

This definition satisfies the corresponding SCL

$$\frac{\delta V}{\Delta t}=u_{g,e}-u_{g,w}, \tag{9}$$

since

$$\frac{\delta V_G}{\Delta t}=\frac{\delta V_e-\delta V_w}{\Delta t}=\frac{\delta x_e-\delta x_w}{\Delta t}, \tag{10}$$

$$\frac{\delta V_{SCL}}{\Delta t}=u_{g,e}-u_{g,w}=\frac{\delta x_e}{\Delta t}-\frac{\delta x_w}{\Delta t}, \tag{11}$$

i.e. $\delta V_G/\Delta t=\delta V_{SCL}/\Delta t$. By convention, the increments like δx_e are calculated as the difference between the new and the old positions.

Most finite volume calculations with moving grids have been done on two-dimensional Cartesian grids with only one set of moving grid lines.[5,6,10,11] They all used (explicitly or implicitly) equation (8) for the grid velocity. Since this is a one-dimensional case as far as the grid movement is concerned, they did not encounter problems related to the SCL. This is probably the reason why the significance of the SCL was not recognized.

Two-dimensional Cartesian grid

In the case of a two-dimensional Cartesian grid the discretized SCL equation reads (see Figure 5)

$$\frac{V^{\mathrm{n}}-V^{\mathrm{o}}}{\Delta t}=(u_{\mathrm{g,e}}-u_{\mathrm{g,w}})\Delta y^{\mathrm{n}}+(v_{\mathrm{g,n}}-v_{\mathrm{g,s}})\Delta x^{\mathrm{n}}. \tag{12}$$

The obvious way of calculating the grid velocities is to use expressions like (8), which may represent the *exact* velocity of the grid line. However, if both sets of grid lines move, this definition of grid velocity will not, in general, satisfy the SCL. For the finite volume procedure considered here, the error introduced is

$$\frac{\delta V_{\varepsilon}}{\Delta t}=(u_{\mathrm{g,e}}-u_{\mathrm{g,w}})(v_{\mathrm{g,n}}-v_{\mathrm{g,s}})\Delta t. \tag{13}$$

The 'missing' volume δV_{ε} is shown in Figure 3 as a shaded area for a few possible situations.

$\rho\,\delta V_{\varepsilon}/\Delta t$ represents an artificial mass source in the continuity equation (3) and, if it is significant compared with the actual change $\rho\,\delta V/\Delta t$, erroneous results will be obtained, i.e. mass conservation will not be assured. In other words, if the grid velocities are calculated from equation (8), then one has to ensure that the ratio $\delta V_{\varepsilon}/\delta V$ is small:

$$\varepsilon=\frac{(u_{\mathrm{g,e}}-u_{\mathrm{g,w}})(v_{\mathrm{g,n}}-v_{\mathrm{g,s}})\Delta t}{(u_{\mathrm{g,e}}-u_{\mathrm{g,w}})\Delta y^{\mathrm{n}}+(v_{\mathrm{g,n}}-v_{\mathrm{g,s}})\Delta x^{\mathrm{n}}}. \tag{14}$$

From equation (14) it is obvious that ε is equal to zero, i.e. no error is introduced, when either (i) one set of grid lines does not move or (ii) all members of one set of grid lines move at the same velocity. In any other case ε is not zero but is directly proportional to the time step, grid velocities and grid spacing. For fixed boundary movement and time step Δt, as the grid is refined ε will remain unchanged. Thus the only way to reduce ε for given grid velocities is to reduce the time step size Δt. This is schematically demonstrated in Figure 4. Therefore the non-satisfying of the SCL imposes an additional constraint on the time step size, which in some cases may be severer than the temporal discretization accuracy constraint. This will be demonstrated on test calculations in the next section.

If the rate of change of the cell volume is decomposed into four terms corresponding to the four cell faces (cf. Figure 5), i.e.

$$\frac{\delta V}{\Delta t}=\frac{\sum_i \delta V_i}{\Delta t}, \qquad i=\mathrm{e, w, n, s}, \tag{15}$$

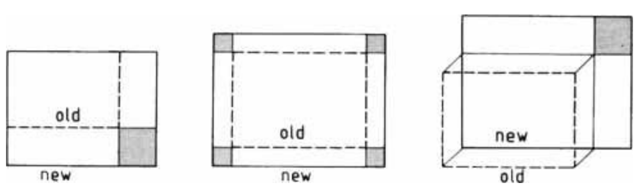

Figure 3. Possible changes of a control volume in time Δt

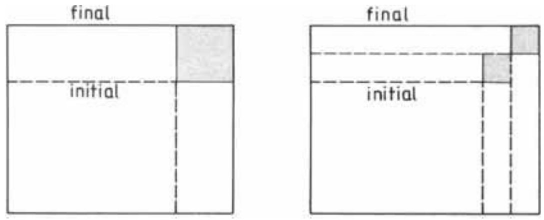

Figure 4. Reduction of SCL error by reducing the time step

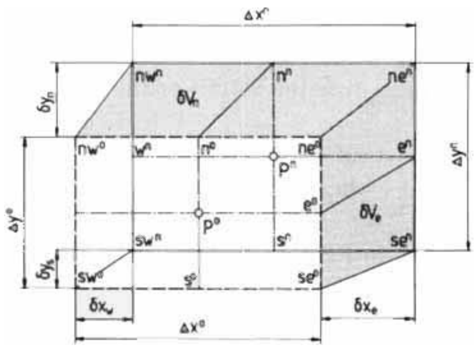

Figure 5. Control volume in two-dimensional moving Cartesian grid at two time levels

then the following definition of the grid velocity components,

$$u_{g,e}=\frac{\Delta y^{o}+\Delta y^{n}}{2\Delta y^{n}}\frac{\delta x_{e}}{\Delta t},$$
$$v_{g,n}=\frac{\Delta x^{o}+\Delta x^{n}}{2\Delta x^{n}}\frac{\delta y_{n}}{\Delta t},\tag{16}$$

guarantees satisfaction of the corresponding SCL (equation (12)) exactly, since

$$\frac{(\delta V_{e})_{G}}{\Delta t}=\frac{\Delta y^{o}+\Delta y^{n}}{2}\frac{\delta x_{e}}{\Delta t},\tag{17}$$

$$\frac{(\delta V_{e})_{SCL}}{\Delta t}=u_{g,e}\Delta y^{n}=\frac{\Delta y^{o}+\Delta y^{n}}{2\Delta y^{n}}\frac{\delta x_{e}}{\Delta t}\Delta y^{n},\tag{18}$$

i.e. $(\delta V_{e})_{G}/\Delta t=(\delta V_{e})_{SCL}/\Delta t$.

Clearly equation (16) reduces to equation (8) if $\Delta y^{o}=\Delta y^{n}$, i.e. if $\delta y_{s}=\delta y_{n}$ (see Figure 5). Note, however, that equations (16) are not the only way of defining grid velocities which automatically satisfy the SCL; other, less obvious, definitions are also possible, but this choice seems the simplest.

Two-dimensional arbitrary grid

In the case of an arbitrary two-dimensional moving grid (see Figure 6) the control volumes at each time level are quadrilaterals defined by four vertices, which are connected by straight lines. The surface vectors and grid velocities have the following components, e.g.

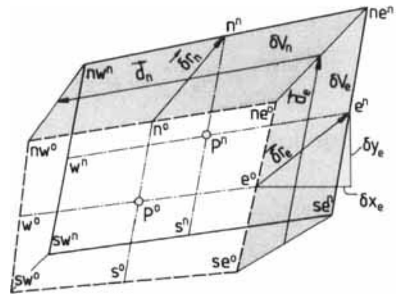

Figure 6. Control volume in two-dimensional moving general grid at two time levels

$$\begin{aligned} \mathbf{S}_e &= (S_x, S_y)_e = (y_{ne} - y_{se}, \ -x_{ne} + x_{se}), \\ \mathbf{S}_n &= (S_x, S_y)_n = (-y_{ne} + y_{nw}, \ x_{ne} - x_{nw}), \\ \mathbf{v}_{g,e} &= (u_g, v_g)_e, \qquad \mathbf{v}_{g,n} = (u_g, v_g)_n. \end{aligned} \tag{19}$$

The definition (16) of the grid velocity components can be generalized to

$$\begin{aligned} u_{g,e} &= \frac{S^o_{x,e} + S^n_{x,e}}{2S^n_{x,e}} \frac{\delta x_e}{\Delta t}, \qquad v_{g,e} = \frac{S^o_{y,e} + S^n_{y,e}}{2S^n_{y,e}} \frac{\delta y_e}{\Delta t}, \\ u_{g,n} &= \frac{S^o_{x,n} + S^n_{x,n}}{2S^n_{x,n}} \frac{\delta x_n}{\Delta t}, \qquad v_{g,n} = \frac{S^o_{y,n} + S^n_{y,n}}{2S^n_{y,n}} \frac{\delta y_n}{\Delta t}, \end{aligned} \tag{20}$$

where δx_e, δy_e, etc. are the components of $\boldsymbol{\delta}\mathbf{r}_e$, etc. (see Figure 6), e.g.

$$\begin{aligned} \boldsymbol{\delta}\mathbf{r}_e &= \mathbf{r}^n_e - \mathbf{r}^o_e = (\delta x_e, \delta y_e) = (x^n_e - x^o_e, \ y^n_e - y^o_e), \\ \boldsymbol{\delta}\mathbf{r}_n &= \mathbf{r}^n_n - \mathbf{r}^o_n = (\delta x_n, \delta y_n) = (x^n_n - x^o_n, \ y^n_n - y^o_n). \end{aligned} \tag{21}$$

The corresponding SCL equation (6) can be rewritten in this case as

$$\frac{\delta V}{\Delta t} = \frac{\delta V_e + \delta V_w + \delta V_n + \delta V_s}{\Delta t} = (\mathbf{v}_g \cdot \mathbf{S}^n)_e + (\mathbf{v}_g \cdot \mathbf{S}^n)_w + (\mathbf{v}_g \cdot \mathbf{S}^n)_n + (\mathbf{v}_g \cdot \mathbf{S}^n)_s. \tag{22}$$

It will now be shown, using as an example the 'e' cell face, that the grid velocities defined by equation (20) do satisfy this SCL equation.

The area swept by the 'e' cell face during time Δt is

$$\frac{(\delta V_e)_G}{\Delta t} = \frac{1}{\Delta t} (\boldsymbol{\delta}\mathbf{r}_e \times \mathbf{d}_e), \tag{23}$$

where (see Figure 6)

$$\mathbf{d}_e = \tfrac{1}{2}(x^n_{ne} + x^o_{ne} - x^n_{se} - x^o_{se}, \ y^n_{ne} + y^o_{ne} - y^n_{se} - y^o_{se}) \tag{24}$$

and where (x^n_{se}, y^n_{se}) are Cartesian co-ordinates of the vertex se^n, for example. It follows from equation (23) that

$$\frac{(\delta V_e)_G}{\Delta t} = \frac{1}{2\Delta t} [\delta x_e (y^o_{ne} + y^n_{ne} - y^o_{se} - y^n_{se}) - \delta y_e (x^o_{ne} + x^n_{ne} - x^o_{se} - x^n_{se})]. \tag{25}$$

On the other hand, from the SCL (equation (22)) we have

$$\frac{(\delta V_e)_{SCL}}{\Delta t} = (u_g S_x^n + v_g S_y^n)_e, \tag{26}$$

and from equations (19) and (20) it follows further that

$$\frac{(\delta V_e)_{SCL}}{\Delta t} = \frac{1}{2\Delta t}[\delta x_e(y_{ne}^o + y_{ne}^n - y_{se}^o - y_{se}^n) - \delta y_e(x_{ne}^o + x_{ne}^n - x_{se}^o - x_{se}^n)]. \tag{27}$$

Equations (25) and (27) are identical, implying that the grid velocities from equations (20) satisfy the SCL. Note again that equations (20) are not unique, but rather the most logical way of calculating grid velocities such that they satisfy the SCL.

The methodology presented above can be readily extended to three-dimensional grids, both Cartesian and arbitrary. For the sake of brevity, this exercise is omitted here.

An alternative way of incorporating the SCL in finite volume methods, which avoids the need to explicitly calculate the grid velocities and thus may especially be attractive in three-dimensional applications, is presented in the Appendix.

4. TESTING OF THE METHOD

The simplest test case is the solution of the mass and momentum conservation equations for a stationary incompressible fluid on a moving grid. With the initial condition set to the exact field (zero velocity and pressure), no fluid motion should result when the SCL is satisfied in the calculations. In order to prove that the proposed method of calculating the grid velocities does, and also to examine the errors introduced by using grid velocities which do not satisfy the SCL, several test cases were set up.

In Case 1 an orthogonal Cartesian and in Case 2 a non-orthogonal grid of 10×10 control volumes (CV) is caused to move from a non-uniform initial position to a uniformly spaced grid at the time $T = 1$ s (see Figures 7 and 8). The grid movement is assumed linear, i.e. each grid line has constant (but different) velocity, except the boundary and centre lines which are fixed. The problem is thus symmetric, the grid velocities being positive on one and negative on the other side of the centre line.

The initial conditions at $t = 0$ were zero velocities and pressure all over the field, the fluid having a density of 1 and a viscosity of 0·01. The dimensions of the solution domain were 10 and 2 in the x, y directions respectively, resulting in maximum grid velocities $u_{g,max} = 0{\cdot}9$ and $v_{g,max} = 0{\cdot}18$.

The problem was solved by a finite volume procedure[9] which incorporates the SIMPLE algorithm for velocity–pressure coupling.[12] Under-relaxation factors of 0·8 for velocities and 0·2 for pressure were used in all calculations (see Perić *et al.*[13] for details). For each case two subcases were run: one with grid velocities specified according to equation (20) and one employing equation (8). Calculations were performed with two time steps: $\Delta t^* = \Delta t/T = 1$ (i.e. the grid moves from its initial to its final position in one time step) and $\Delta t^* = 0{\cdot}2$ (the same grid movement occurs over five time steps).

When the grid velocities are calculated according to equation (20), the exact solution is obtained in one iteration in both cases and both time steps. No numerical errors are introduced when the grid velocities satisfy the SCL, and thus the time step may be as large as allowed by the temporal discretization error. (In the above case an arbitrarily large time step may be used, since the solution does not change with time.)

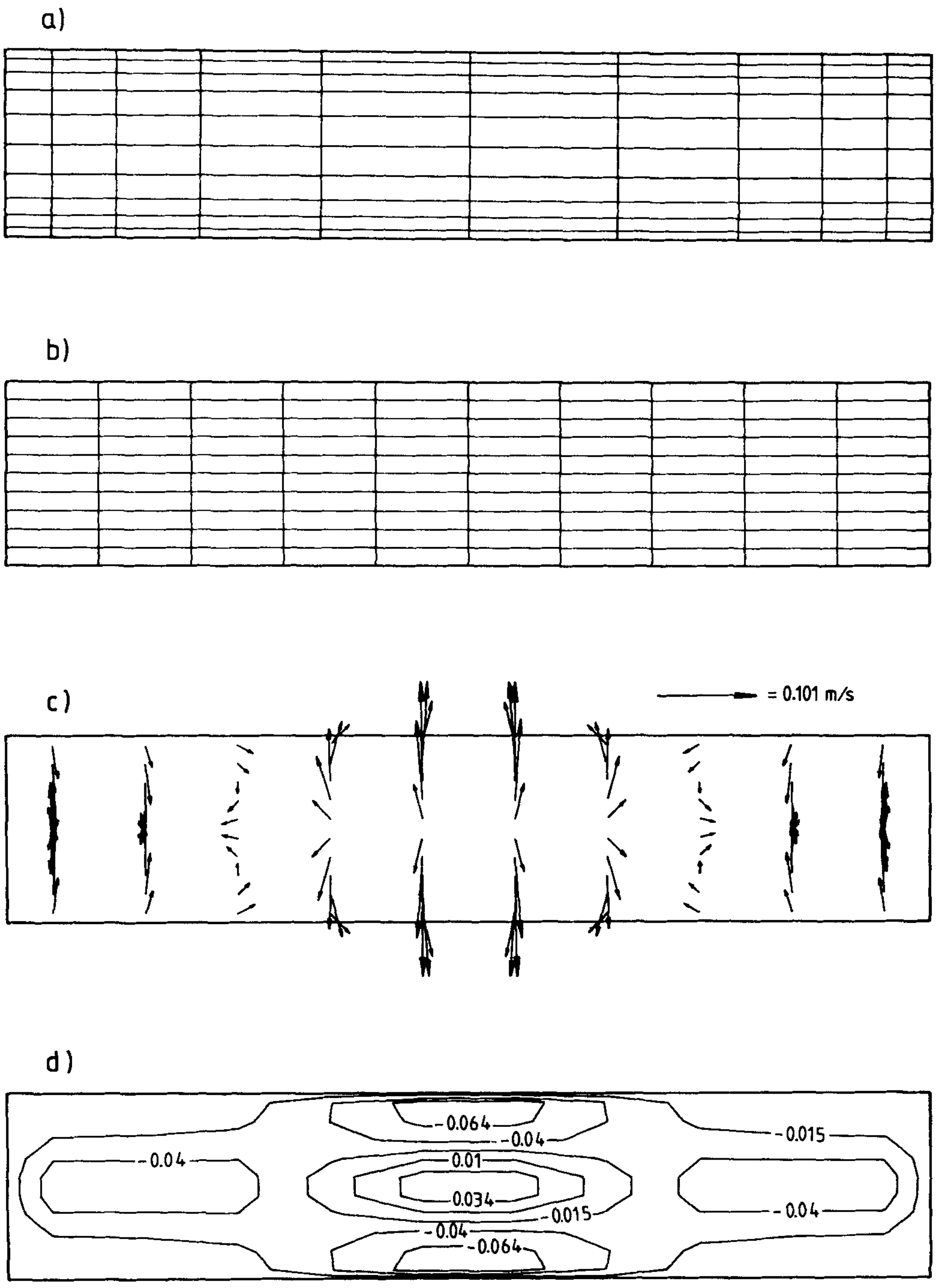

Figure 7. Case 1. initial (a) and final (b) grid, predicted velocity vectors (c) and pressure contours (d) at $t = T$, obtained using $\Delta t^* = 1$

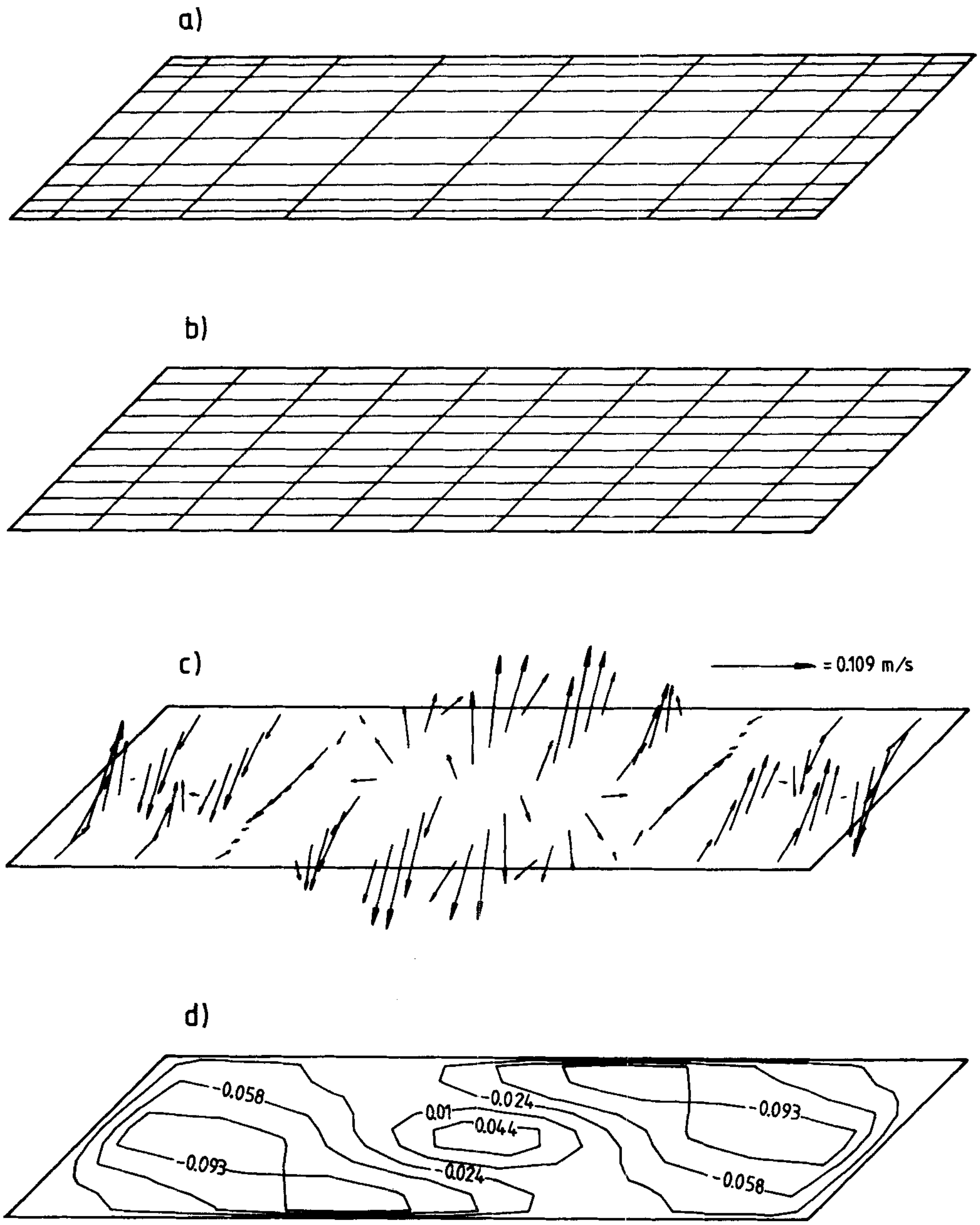

Figure 8. Case 2. initial (a) and final (b) grid, predicted velocity vectors (c) and pressure contours (d) at $t = T$, obtained using $\Delta t^* = 1$

When, however, the grid velocities are calculated according to equation (8) (which are the *exact* grid velocities), the exact initial solution is destroyed by the artificial mass sources introduced. The maximum errors in u, v and p at the end of each time step are shown in Tables I and II for these two cases respectively. The resulting velocity vectors and pressure contours at $t = T$ (grid in its final position) are presented in Figures 7 and 8.

Table I. Maximum errors in Case 1

	$\Delta t^*=1$	$\Delta t^*=0{\cdot}2$				
t	1·0	0·2	0·4	0·6	0·8	1·0
u_{max}	0·0196	0·0054	0·0039	0·0033	0·0034	0·0039
u_{min}	−0·0196	−0·0054	−0·0039	−0·0033	−0·0034	−0·0039
v_{max}	0·1009	0·0261	0·0243	0·0218	0·0196	0·0211
v_{min}	−0·1009	−0·0261	−0·0243	−0·0218	−0·0196	−0·0211
p_{max}	0·0467	0·0009	0·0020	0·0039	0·0052	0·0067
p_{min}	−0·0765	−0·0840	−0·0085	−0·0046	−0·0043	−0·0045

Table II. Maximum errors in Case 2

	$\Delta t^*=1$	$\Delta t^*=0{\cdot}2$				
t	1·0	0·2	0·4	0·6	0·8	1·0
u_{max}	0·0426	0·0188	0·0187	0·0165	0·0142	0·0122
u_{min}	−0·0426	−0·0188	−0·0187	−0·0165	−0·0142	−0·0122
v_{max}	0·1063	0·0268	0·0258	0·0235	0·0214	0·0226
v_{min}	−0·1063	−0·0268	−0·0258	−0·0235	−0·0214	−0·0226
p_{max}	0·0615	0·0000	0·0049	0·0049	0·0064	0·0089
p_{min}	−0·1098	−0·1545	−0·0229	−0·0088	−0·0078	−0·0079

Tables I and II show that the maximum error (in this case for v) is reduced by a factor of approximately five when the time step is reduced by this factor. This is consistent with the error analysis presented in the previous section. For the non-orthogonal grid the errors in v were about the same, and those in u and p about twice as large as for the orthogonal grid.

Figures 7 and 8 indicate that, as a result of not satisfying the SCL, the solution process converges to an unrealistic flow field (between 30 and 10 iterations per time step were needed). The errors are largest in regions where largest changes in control volume size occur, as expected. When the time step is reduced, the errors are also reduced in magnitude, but the pattern remains the same.

Cases 3, 4 and 5 involve calculations with the 'exact' grid velocities (equation (8)) on 20 × 20 CV grids. In Case 3 the grid covers the same domain as in Case 1, but Δx and Δy are approximately halved. The time interval T is the same (1 s), so the velocities of corresponding grid lines are also the same. However, since there are more grid lines than in Case 1, the differences in neighbour grid line velocities are about halved. According to expression (14), one should expect errors of the same order as in Case 1 to occur (Δt the same, Δx_i and Δu_g halved). Table III shows errors for three different time steps Δt^*: 1, 0·5 and 0·1. For the same time step the errors are nearly equal to those presented in Table I. Reduction of the time step again resulted in a corresponding reduction of the error.

Table III. Maximum errors in Case 3

	$\Delta t^*=1$	$\Delta t^*=0.5$		$\Delta t^*=0.1$		
t	1·0	0·5	1·0	0·1	0·5	1·0
u_{max}	0·0242	0·0126	0·0111	0·0056	0·0037	0·0028
u_{min}	−0·0242	−0·0126	−0·0111	−0·0056	−0·0037	−0·0028
v_{max}	0·1064	0·0693	0·0535	0·0176	0·0132	0·0107
v_{min}	−0·1064	−0·0693	−0·0535	−0·0176	−0·0132	−0·0107
p_{max}	0·0452	0·0004	0·0165	0·0016	0·0024	0·0034
p_{min}	−0·0802	−0·0814	−0·0116	−0·1118	−0·0026	−0·0020

Case 4 uses the same grid as Case 3, but with twice the grid velocities: the time T is reduced to 0·5 s. Thus the velocity differences Δu_g in equation (14) are doubled; however, Δt is halved, so ε is unchanged, i.e. the error *relative to grid velocity* is the same. However, $\delta V_g/\Delta t$ is twice that of the previous case, so doubled absolute errors are expected. Table IV shows that the results accord with these expectations. The velocity vectors and pressure contours show the same patterns (but with magnitudes twice as high) as already seen in Figure 7 for Case 1.

The same effect is achieved in Case 5 by leaving the time interval the same as in Case 3 and doubling the size of the solution domain, so that the grid velocities are again doubled. The results of calculations are presented in Table V, and they are indeed almost identical to those of Table IV.

5. CONCLUSIONS

The need to satisfy the SCL, which relates the rate of change of the control volume to the velocity of its boundary, in fluid flow calculations with moving grids has been demonstrated. It is shown that for incompressible fluid flow the SCL is essential to mass conservation; and not satisfying it means the introduction of artificial mass sources.

In order to satisfy the SCL exactly in finite volume calculations, it is sufficient to calculate the grid velocities from the known new and old time level locations of grid lines in an appropriate

Table IV. Maximum errors in Case 4

	$\Delta t^*=1$	$\Delta t^*=0.2$				
t	0·5	0·1	0·2	0·3	0·4	0·5
u_{max}	0·0510	0·0177	0·0143	0·0114	0·0093	0·0089
u_{min}	−0·0510	−0·0177	−0·0143	−0·0114	−0·0093	−0·0089
v_{max}	0·2130	0·0669	0·0584	0·0511	0·0454	0·0429
v_{min}	−0·2130	−0·0669	−0·0584	−0·0511	−0·0454	−0·0429
p_{max}	0·1950	0·0015	0·0249	0·0247	0·0246	0·0291
p_{min}	−0·2760	−0·3890	−0·0163	−0·0102	−0·0089	−0·0092

Table V. Maximum errors in Case 5

	$\Delta t^*=1$	$\Delta t^*=0{\cdot}2$				
t	1·0	0·2	0·4	0·6	0·8	1·0
u_{max}	0·0564	0·0171	0·0139	0·0110	0·0089	0·0090
u_{min}	−0·0564	−0·0171	−0·0139	−0·0110	−0·0089	−0·0090
v_{max}	0·2139	0·0671	0·0588	0·0515	0·0458	0·0429
v_{min}	−0·2139	−0·0671	−0·0588	−0·0515	−0·0458	−0·0429
p_{max}	0·2048	0·0008	0·0296	0·0278	0·0270	0·0302
p_{min}	−0·2505	−0·3764	−0·0092	−0·0053	−0·0057	−0·0067

way. One simple (but not the only possible) way of calculating the grid velocities for two-dimensional Cartesian and general non-orthogonal grids was presented which fulfils the above requirement. Test calculations performed on stagnant fluids and various moving grids verified that the proposed method of grid velocity calculation is correct.

It is also demonstrated on the same test cases that the grid velocities deduced from the law of their motion do not satisfy the SCL and introduce serious errors. The errors are shown to depend on the grid velocities, time step and grid spacing. For problems in which the grid velocities are dictated by the movement of boundaries, the only way of reducing the error is to reduce the time step, since grid refinement does not affect the SCL error. Thus not satisfying the SCL implies an additional constraint on the time step size, which may be severer than the temporal discretization accuracy requirement. When, on the other hand, the grid velocities satisfy the SCL, no additional error is introduced and accurate results are obtained for all time steps consistent with temporal accuracy.

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APPENDIX

In this Appendix an alternative way of incorporating the SCL into a finite volume method for the solution of fluid flow and heat transfer conservation equations is presented.

The integral form of the SCL for an arbitrary control volume (see Figure 1) for a fully implicit time differencing scheme is given by equation (6):

$$\frac{\delta V}{\Delta t}=\sum_i \mathbf{v}_{gi}\cdot\mathbf{S}_i^{n},\quad i=\mathrm{e, w, n, s}. \tag{28}$$

The total rate of change of the control volume may be decomposed in the following way (see Figure 5 or 6 for example):

$$\frac{\delta V}{\Delta t}=\frac{\sum_i \delta V_i}{\Delta t}, \tag{29}$$

where each component is

$$\frac{\delta V_i}{\Delta t} = \mathbf{v}_{gi} \cdot \mathbf{S}_i^n. \tag{30}$$

The convection term in the conservation equation for an arbitrary variable ψ (where ψ stands for ρ, u, v or ϕ, cf. equations (1)) may be discretized as (cf. equation (6))

$$\sum_i (\rho\psi)_i (\mathbf{v} - \mathbf{v}_g)_i \cdot \mathbf{S}_i^n = \sum_i (\rho\psi)_i (\mathbf{v}_i \cdot \mathbf{S}_i^n - \mathbf{v}_{gi} \cdot \mathbf{S}_i^n)$$
$$= \sum_i (\rho\psi)_i \left(\mathbf{v}_i \cdot \mathbf{S}_i^n - \frac{\delta V_i}{\Delta t} \right). \tag{31}$$

The rate of change of the cell volume is calculated from the known grid positions (cf. equations (17) and (23)), i.e.

$$\frac{\delta V_i}{\Delta t} = \frac{(\delta V_i)_G}{\Delta t}, \quad i = \mathrm{e, w, n, s}. \tag{32}$$

This approach is equivalent to that presented in Section 3, since the proposed method of calculating the grid velocities stems from the same requirement. However, it may be more convenient since it does not require definition of the grid velocities. This applies especially to three-dimensional cases, where at each cell face three components of the grid velocity would have to be calculated, as opposed to the calculation of a single rate of change of the cell volume in equation (31).

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